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DYNAMICS OF A SYSTEM OF COUPLED NONLINEAR OSCILLATORS WITH PARTIAL ENERGY DISSIPATION

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Abstract. We study the dynamics of the 3-DoF system with bistable nonlinear energy sink absorber, connected to the primary 2-DoF system. With the assumption that the mass of the absorber is small enough we study the local behaviour of the system in the neighbourhood of the stationary points. Under condition that parameters of the mechanical system are uncertain, the analytical approximations of eigenvalues of the linearized system are presented. These approximations are used to determine the characteristics of absorber (damping coefficient and negative linear stiffness component), which guarantee the optimal damping rate of the responses of the main system to perturbations. The theoretical results are compared with numerical experiments. The dependence between linear and nonlinear stiffnesses of the absorber is discussed.

1 INTRODUCTION

Problems associated with undesirable vibrations are encountered in many applied tasks in mechanical engineering, construction, aerospace engineering, biomechanics, etc. For various reasons, a structure may encounter sources of excitation, which are not predicted for the project. To increase reliability, design engineers seek a simple, low-cost, and efficient solution. In many cases, the dynamic vibration absorbers (DVA) are used that meet these requirements. Dynamic vibration absorbers or tuned mass dampers are small elements locally attached to the structure, designed to dissipate the excessive vibration energy.

One of the important goals of vibration control is to create a frequency zone in which resonance (or a spectral gap around an inconvenient frequency) cannot occur, by connecting a vibration absorber. In this case, usually DVA parameters are determined in accordance with the eigenvector of the unstable oscillation mode, which ensures the spatial distribution of the oscillation energy within one oscillation mode. The disadvantage of such a single-mode approach is that it does not take into account the influence of adjacent oscillation modes, which may become important in some circumstances.

In order to solve the problem with a narrow band TMD, remaining within the framework of passive interference suppression, with using only one device, many researchers studied the effect of additional nonlinearities in the absorber [1 – 10], seeking to allow a resonator to resonate at more than one frequency. This led to the development of the concept of nonlinear energy sink (NES), which we can represent schematically in the form of a small mass, connected with the primary system by essential nonlinearity. On today, a number of NES constructions have been proposed, such as cubic nonlinearity [11], a vibro-impact device [12, 13], an eccentric rotator [14], and a tuned pendulum [15]. Also, more recently, a bistable NES (BNES) has been proposed, consisting of a small mass connected with the primary system by a spring with both cubic nonlinear and negative linear components [16, 17].

2 DESCRIPTION OF THE SYSTEM AND STABILITY ANALYSIS

We consider the 3-Degrees-of-Freedom mechanical system which is schematically presented on fig.1. It consists of two masses connected to each other and to fixed supports by springs (linear) and a bistable absorber connected to one of them.

The motion equations of the mechanical system considered are

$$\begin{aligned} m_1\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_a) + k_1x_1 + k_2(x_1 - x_2) - k_a^{lin}(x_1 - x_a) + k_a^{nonlin}(x_1 - x_a)^3 = 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3x_2 = 0 \\ m_a\ddot{x}_a + c(\dot{x}_a - \dot{x}_1) + k_a^{lin}(x_1 - x_a) - k_a^{nonlin}(x_1 - x_a)^3 = 0. \end{aligned} \tag{1}$$

Here x_1 and x_2 refer to the displacements of the primary 2-DoF system, while x_a refers to the displacement of the bistable absorber; m_1, m_2 ($m_2 \leq m_1$) are masses of the bodies, and $k_j (j = 1, 2, 3)-$ the stiffnesses of the springs; $c, k_a^{lin}, k_a^{nonlin}$ are the absorber damping coefficient, negative linear spring coefficient and cubic (positive) spring coefficient respectively. The overdots denote the differentiation on time. It is supposed that parameters of main system are given, parameters of absorber are tunable, and the mass of the absorber m_a is much less then m_1.

It is easy to see that the ODE system (1) has three stationary points: the origin (unstable one) and two points $(0, 0, \pm\sqrt{k_a^{lin}/k_a^{nonlin}})$ on the axis Ox_a. Let us introduce the perturbation

$$x_a = \sqrt{k_a^{lin}/k_a^{nonlin}} + \widetilde{x}_a,$$

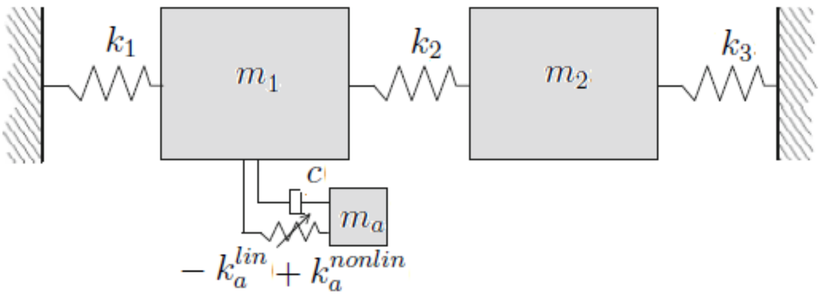


Figure 1: Three-degree-of-freedom system consisting of two coupled linear oscillators and a bistable absorber connected to one of them

and dimensionless parameters and time with formulas

$$\beta = \frac{m_2}{m_1},\ \mu = \frac{m_a}{m_1},\ \varkappa_1 = \frac{k_1}{m_1},\ \varkappa_2 = \frac{k_2}{m_2},\ \varkappa_3 = \frac{k_3}{m_2},$$
$$(\omega^2)_{1,2} = \frac{1}{2}(p \pm \sqrt{p^2 - 4q}),\ p = \varkappa_1 + (1+\beta)\varkappa_2 + \varkappa_3,\ q = \varkappa_1(\varkappa_2 + \varkappa_3) + \beta\varkappa_2\varkappa_3, \quad (2)$$
$$h = \frac{c}{m_a\omega_2},\ \varkappa = \frac{2k_a^{lin}}{m_a\omega_2^2},\ \tau = \omega_2 t.$$

Here ω_1, ω_2 are the natural frequencies of the main system.

The linearized system in the vicinities of stationary points is associated with the following $\lambda-$ matrix

$$\begin{pmatrix} \omega_2^2\lambda^2 + \varkappa_1 + \beta\varkappa_2 & -\beta\varkappa_2 & -\mu\omega_2^2(h\lambda + \varkappa) \\ -\varkappa_2 & \omega_2^2\lambda^2 + \varkappa_2 + \varkappa_3 & 0 \\ -\mu\omega_2^2(h\lambda + \varkappa) & 0 & \omega_2^2(\lambda^2 + h\lambda + \varkappa) \end{pmatrix}. \quad (3)$$

Accordingly, the characteristic polynomial is as follows

$$f(\lambda) = \omega_2^4\lambda^6 + h\omega_2^4(1+\mu)\lambda^5 + \omega_2^2[\varkappa_1 + \varkappa_2(1+\beta) + \varkappa_3 + \varkappa(1+\mu)]\lambda^4 +$$
$$+h\omega_2^4[\varkappa_1 + \varkappa_2(1+\beta+\mu)]\lambda^3 + \omega_2^2\{\varkappa_1(\varkappa_2+\varkappa_3) + \beta\varkappa_2\varkappa_3 + \varkappa[\varkappa_1 + \varkappa_2(1+\beta+\mu) + \quad (4)$$
$$+\varkappa_3(1+\mu)]\}\lambda^2 + h(\varkappa_1\varkappa_2 + \varkappa_1\varkappa_3 + \beta\varkappa_2\varkappa_3)\lambda + \varkappa(\varkappa_1\varkappa_2 + \varkappa_1\varkappa_3 + \beta\varkappa_2\varkappa_3).$$

For any set of parameters of the system considered all coefficients of the polynomial $f(\lambda)$ are strictly positive, as well as determinants

$$\Delta_3 = \mu h^2[\varkappa_1^2 + \beta\varkappa_2^2(1+\beta+\mu) + 2\beta\varkappa_1\varkappa_2],$$
$$\Delta_5 = \mu^2 h^3\beta(\varkappa_2^2\varkappa_1\varkappa_2 + \varkappa_1\varkappa_3 + \beta\varkappa_2\varkappa_3)^2. \quad (5)$$

Then, according to Lienard – Chipart criterion [18], all eigenvalues of linearized system have negative real parts, i.e. two non-origin stationary points are stable focuses (spirals).

3 ESTIMATION OF EIGENVALUES AND TUNING THE ABSORBER

Now we shall count μ as small parameter. In origin case $\mu = 0$ the polynomial $f(\lambda)$ has six different roots:

$$\pm\imath, \pm\imath\frac{\omega_1}{\omega_2}, -\frac{1}{2}(h \pm \sqrt{h^2 - 4\varkappa}).$$

Consequently the eigenvalues of linearized system are analytical functions on μ and may be presented as taylor expansions

$$\lambda_j = \lambda_{j0} + \mu\lambda_{j1} + \mu^2\lambda_{j2} + \cdots, j = \overline{1,6} \tag{6}$$

and the coefficients λ_{js} are calculated sequentially by substitution expression (6) in (4). Taking into account that the corresponding expressions in general case are very bulky, in present paper we limit ourselves with "canonical" case: $m_2 = m_1, k_3 = k_2 = k_1$. Hence, $\beta = 1, \omega_1 = \sqrt{3}\omega_2$. Thus we have

$$\lambda_{11} = -\frac{1}{4}\frac{h + \imath(h^2 + \varkappa^2 - \varkappa)}{h^2 + (\varkappa - 1)^2}, \lambda_{31} = -\frac{\sqrt{3}}{4}\frac{3\sqrt{3}h + \imath(3h^2 + \varkappa^2 - 3\varkappa)}{3h^2 + (3 - \varkappa)^2},$$
$$Re\lambda_{12} = \frac{h}{8}\frac{h^4 + h^2(3\varkappa^2 - 3\varkappa - 1) + \varkappa(2\varkappa^3 - 3\varkappa^2 + 1)}{[2h^2 + (\varkappa - 1)^2]^3}, \cdots. \tag{7}$$

The damping rate of the oscillations of the system is governed by maximal Lyapunov exponent value. In rather small vicinity of stationary point this value is close to the $maxRe\lambda_j$ $(j = \overline{1,6})$, i.e. in first approximation the biggest of values $Re\lambda_{11}, Re\lambda_{31}$ is responsible for it.

Thus, for tuning the absorber we suggest the minimization of function $z(h, \varkappa) = max(Re\lambda_{11}, Re\lambda_{31})$. There are two possibilities here:

A) The surface

$$z_1(h, \varkappa) = [h^2 + (\varkappa - 1)^2]^{-1}$$

intersects the surface

$$z_2(h, \varkappa) = 9[3h^2 + (3 - \varkappa)^2]^{-1};$$

B) there is no intersection.

Case A) leads to relation

$$h^2 = \frac{2}{3}\varkappa(3 - 2\varkappa). \tag{8}$$

Now, if the value of z is maximized on the set (8), then any other point of the both surfaces is situated below and gives the bigger value to $Re\lambda$. Substituting h from (8) to z_1 (or z_2, no matter) we have the function of one argument

$$\psi(\varkappa) = \frac{\varkappa(3 - 2\varkappa)}{(3 - \varkappa^2)^2} \ (\varkappa > \frac{3}{2}).$$

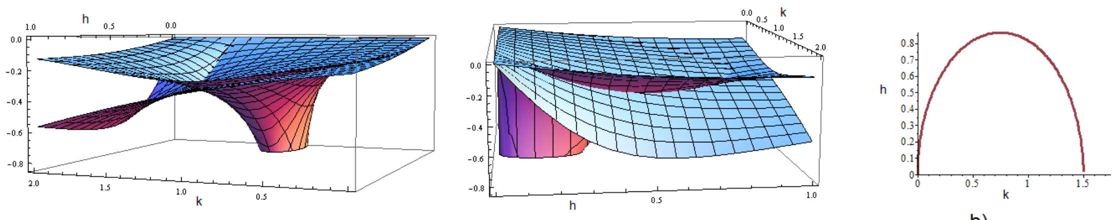


Figure 2: The intersection of the surfaces $z_1(h, \varkappa)$ and $z_2(h, \varkappa)$

Its derivative is equal to zero as

$$9 - 12\varkappa + 9\varkappa^2 - 4\varkappa^3 = 0.$$

The roots of the last equation are $\approx 1.2797; 0.4852 \pm 1.2341\imath$, and the real one gives the maximal value for $\psi(\varkappa)$. Consequently, $h \approx 0.6131, Re\lambda_{11} \approx -0.3375\mu$.

The case B) is characterized by condition $\varkappa \geq 1.5$, and $Re\lambda_{11} > Re\lambda_{31}$. For any value of h the function $z_1(h, \varkappa)$ is decreasing on argument $\varkappa$, and the least possible value of $\varkappa$ is the choice, and $h = \varkappa - 1$. Then $\varkappa = 1.5$, $h = 0.5$, and $Re\lambda_{11} = -0.25\mu$. Definitely, this value is worse comparatively with -0.3375μ, so the choice of case A) is optimal.

Remark. It should be noted that the next terms of expansion of $Re\lambda$ are positive and decrease slightly the initial value. For instance, with $\mu = 0.04, \varkappa = 1.2797, h = 0.6131$ the calculated value of $maxRe\lambda$ is -0.0135. At the same time the roots of $f(\lambda)$ are:

$$-0.01371 \pm 1.7365\imath, -0.01263 \pm 0.9828\imath, -0.2925 \pm 1.1101\imath.$$

So, there is about 7 percent error. This fact must be taking into account while tuning the absorber. The calculated values of h and $\varkappa$ may be taken as the initial point, and numerical scrolling the neighborhood of this point will lead to the best effect.

4 Numerical simulations and discussion

We have simulated the motion of the system by integrating the equations (1) (in dimensionless parameters) with small enough initial values for primary system – $x_1(0), x_2(0), x_1'(0), x_2'(0)$ (prime means the differentiation on time τ). Generally, the results are consistent with theoretical expectations, however few things should be noted. One of them is the influence of nonlinear stiffness of the absorber on the responses of the system. Numerical experiments show that its value must be small enough to prevent the mitigation of oscillations (see also [19]). With this condition the typical view of phase trajectories is illustrated on fig.3 ($\mu = 0.04, h = 0.7, \varkappa = 1.4, k_a^{nonlin} = 0.004$). Displacements of the primary masses are bounded (fig.3a), and slowly on τ tend to zero (fig. 3b). Initial perturbation of absorber is not so important - it goes to the vicinity of stationary point very quickly (fig. 3c).

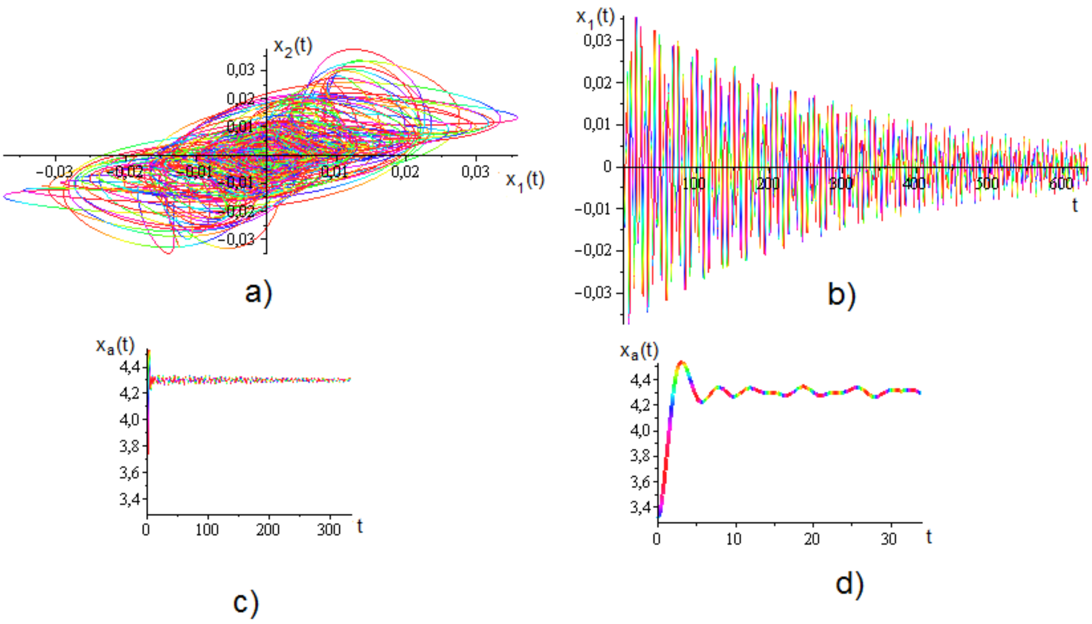


Figure 3: Displacements of the masses of main system and time histories, $\varkappa = 1.4$

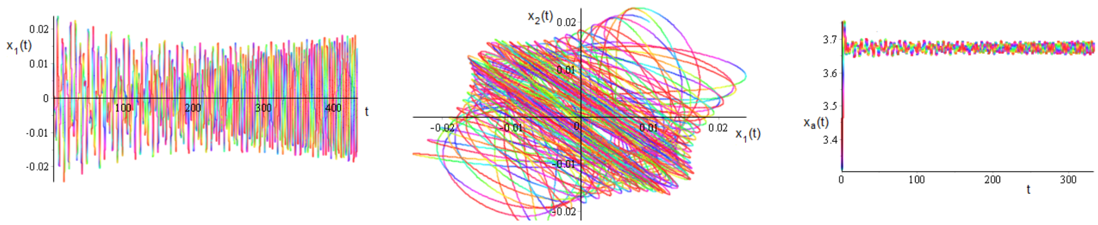
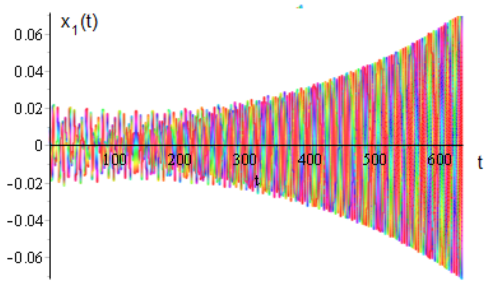


Figure 4: Projection of phase trajectory, $\varkappa = 1.2$

The variation of dimensionless damping coefficient h on interval $[0.5, 1]$ is not very important, while small values (below 0.2) as well as big (more then 1.3) have negative effect on system's behaviour.

The influence of parameter $\varkappa$ on system's dynamics is also remarkable. The calculated value is equal to 1.2797 (1.414 in [19]). The decreasing of this value leads to the following results. The "mediocre" decrease has the dual effect: the responses of the main system becomes weaker (compare with fig.3a), but the amplitude of oscillations decreases very slowly (fig.4b, 4c). The further decrease leads to growth of the responses (and, probably, to chaotic motion). The initial values for all figures were taken identical, so the "spindle" form of trajectory on fig.5 is in discord with its counterparts on fig.3 and fig.4.



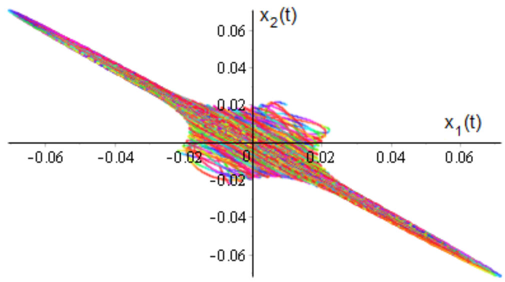


Figure 5: Projection of phase trajectory, $\varkappa = 0.9$

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